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Verhulst-type kinetics driven by white shot noise: Exact solution by direct averaging

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The influence of parametric noise on a class of a growing-rate-type model is studied. The formal time-dependent solution is found analytically for any linearly coupled white noise. The Mellin-Barnes technique is used in order to extend this solution on a whole parameter space in the case of Gaussian white noise and white shot noise. The asymptotic behavior in different regions of parameter space is examined. A digital averaging is done which supports the analytical results. It is found that, in contrast to the deterministic case, the relaxation in the presence of the noise needs not be purely monotonic.

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I. INTRODUCTION

It is now well known that both kinetic and stationary properties of nonlinear macroscopic systems are strongly influenced by the presence of noise (fluctuations), up to the appearance of new types of behavior, such as the stabilization or destabilization of the process, the appearance of noise-induced transitions, among others [1–9]. These changes may depend both quantitatively and qualitatively on the character of the noise present, i.e., on the properties of stochastic processes which describe the noise considered [2,3].

In general, there are two possibilities of studying the influence of stochastic effects on dynamical systems described by equation

$$\dot{x}(t)=f(x,t)+g(x,t)\xi_t \;, \tag{1.1}$$

where ξ_t denotes the noise and x is the time-dependent order parameter, chemical concentration, or other macroscopic characteristics of the system. The first one consists of calculating the probability density $P(x,t)$, usually by solving appropriate differential equations–the Fokker-Planck equation, the master equation, the Smoluchowski equation [1,4]. The observable quantities are then obtained by averaging the appropriate functions of x with $P(x,t)$. Closely related are the so-called decoupling methods [5], where one considers approximate relations between different averaged quantities (correlation functions, reduced probabilities, etc.) by truncating appropriate infinite hierarchies of equations for these quantities. The second group, the so-called direct methods [3,6–9], look for stochastic quantities—like the solution $x(t)$ of stochastic kinetic equation (1.1)—themselves, the averaged (observable) quantities then being calculated by averaging of appropriate functions of $x(t)$ over all realizations of the stochastic process [10].

Most popular in the literature is the approach using the Fokker-Planck equation (FPE), and most specific results concerning the influence of noise on kinetic processes have been obtained by using this formalism. However, so far its usefulness is limited almost exclusively to stationary properties, although there is recently some progress in finding time-dependent solutions of FPE's [8,11]. Moreover, in most cases only the Gaussian white noise (GWN) is considered. The explicit form of FPE for white shot noise (WSN) is also known, but again its practical use is limited to some specific types of WSN only [2,12]. In the case of colored noises the very formulation of proper FPE's is still under debate [13].

In all these aspects, direct methods may be considered as complementary to the FPE. When the solution of Eq. (1.1) is known, time-dependent problems can be treated on equal footing with stationary ones. When averaging is done numerically, different types of the noise—both white and colored—can be simulated with equal ease.

The disadvantage is that the solution (in quadratures) of the ordinary nonautonomous equation (1.1) must be known and it restricts the forms of f and g. The latter condition can be met, in particular, for systems with kinetic equation of the Verhulst type

$$\dot{x}(t)=a(t)x-b(t)x^{\mu+1} \;. \tag{1.2}$$

Such equations can model many different processes and are discussed in the literature frequently [14].

In this paper we propose a version of direct method for treatment of the solution of the kinetic equation (1.2), when the parameter $a(t)$ contains (multiplicative) noise and the deterministic coefficients are assumed to be constant [15]. The method proposed enables analytical treatment of moments $\langle x^n(t)\rangle$ for different types of white noise. The formulas obtained will be used for the determination of parameter regions in which noise destabilizes the kinetic process. Also, conclusions on the asymptotic behavior of the relaxation under the influence of noise can be drawn.

The outline of the paper is as follows. In Sec. II we discuss some basic properties of the considered model, in Sec. III the general formulas for the (transient) moments in the presence of white noise are obtained by direct averaging. Section IV is devoted to the analysis of the influence of some particular types of white noise. In Sec. V the numerical results are presented, and Sec. VI contains final remarks and conclusions. Technical details, related to the use of the Mellin-Barnes representation of the infinite series in Sec. IV, are presented in the Appendixes.

II. THE MODEL–BASIC PROPERTIES

Let the coefficients in Eq. (1.2) be

$$a(t)=a+A\xi_t\ ,\quad b(t)=b\ , \tag{2.1}$$

where ξ_t describes the noise, which will be specified later, $A\neq 0$ and $b\mu>0$. The last condition enables us to consider the stochastic solution

$$x_t=x_0e^{at}\exp(A\mathcal{W}_t)\times\left[1+\mu x_0^{\mu}b\int_0^t ds\, e^{\mu as}\exp(\mu A\mathcal{W}_s)\right]^{-\nu}\ , \tag{2.2}$$

where $\nu=1/\mu$, $t\geq 0$, and

$$\mathcal{W}_u=\int_0^u \xi_s ds \tag{2.3}$$

for $x_0=x_{t=0}$ from the whole domain $(0,+\infty)$, without restrictions on "phase space."

The probabilistic characteristics of the stochastic process x_t may be obtained by the averaging (over all possible trajectories of stochastic process $\mathcal{W}$) of appropriate expressions built of the solution (2.2). For this purpose note that the increments of processes $\mathcal{W}_t$ will be stationary and independent for the (stationary) white noises ξ_t. Note also that for a complete characterization of the Markov process, x_t given by Eq. (2.2) it is sufficient to know the mean value $\langle x_t(a,A,b,\nu,x_0)\rangle$ as a function of its arguments. Namely, (2.2) gives immediately the following "scaling" rule:

$$\langle x_t^{\omega}(a,A,b,\nu,x_0)\rangle=\langle x_t(\omega a,\omega A,\omega b,\omega\nu,x_0^{\omega})\rangle\ . \tag{2.4}$$

Thus the higher (transient) moments [16] of x_t for the system $\{a,A,b,\nu,x_0\}$ [i.e., the physical system characterized by values of parameters a,A,b,ν of Eq. (1.2), and by the initial state x_0] are determined by the first moments of appropriately rescaled systems $\{\omega a,\omega A,\omega b,\omega\nu,x_0^{\omega}\}$.

Such a procedure is useless when the averaging is to be done numerically [6]. Hence the knowledge of analytical averaging formulas seems to be desirable, and such analytical averaging is the main aim of this work. Unfortunately again, such formalism can be constructed only for some types of noise. For other types there remain, so far, numerical computations. Nevertheless, the formalism reported here enables us to obtain several results which are very difficult to get from the FPE method. In this sense the proposed formalism may be understood as complementary to the Fokker-Planck equation. Especially, the main advantage of the present method is that both the stationary and the time-dependent distributions can be obtained with the same amount of work, whereas the time-dependent solutions of the Fokker-Planck equation are practically unknown (except for some of the simpler cases).

The general form of the moments generating function (MGF) of the process $\mathcal{W}$, given by Eq. (2.3) as an integral of white noise, reads

$$\langle\exp(y\mathcal{W}_t)\rangle=e^{\lambda t\varphi(y)}\ . \tag{2.5}$$

In order to prove (2.5) let us define $e^{\phi(t,y)}=\langle\exp(y\mathcal{W}_t)\rangle$. The independence of increments of the process $\mathcal{W}$ enables us to write the following equality: $\langle\exp[y(\mathcal{W}_t-\mathcal{W}_s)]\rangle\langle\exp(y\mathcal{W}_s)\rangle=\langle\exp(y\mathcal{W}_t)\rangle$. Then, using stationarity of increments of $\mathcal{W}$, we obtain $\phi(t-s,y)+\phi(s,y)=\phi(t,y)$, and conclude that ϕ must be a linear function of time.

For GWN ξ_t with the strength λ [i.e., with the autocorrelation $\langle\xi_t\xi_s\rangle=2\lambda\delta(t-s)$], $\mathcal{W}_t$ is the Wiener process and

$$\varphi(y)=y^2\ , \tag{2.6}$$

and for ξ_t being (generalized) WSN [17]:

$$\xi_t=\sum_n \zeta_n\delta(t-t_n)-\lambda\langle\zeta\rangle\quad(\langle\xi_t\rangle=0)\ , \tag{2.7}$$

where t_n are the random points on the time axis given by the Poisson process with parameter λ, and ζ_n are independent random variables with the same arbitrary distribution; function φ satisfies the relation

$$\varphi(y)=\langle\exp y\zeta\rangle-y\langle\zeta\rangle-1\ , \tag{2.8}$$

i.e., it is simply related to MGF of the distribution of ζ, which generates the WSN. It takes the particular form of

$$\varphi(y)=\frac{\beta}{\beta-y}-\frac{y}{\beta}-1 \tag{2.9}$$

for WSN with random weights $\zeta\in(0,\infty)$ given by the probability density $p(\zeta)=\beta e^{-\beta\zeta}$.

III. AVERAGING—GENERAL FORMULAS

As we have seen in the preceding section, to solve our problem, we have to calculate the mean value of (2.2). Although, in the case of GWN it has been done by the method based on binomial expansion of (2.2) (see, for instance, Refs. [9,18]), it is more convenient in the cases considered here to express (2.2) in the form of a Laplace-type integral

$$x_t=\int_0^\infty ds\frac{s^{\nu-1}}{\Gamma(\nu)}\exp[-sx_0^{-\mu}e^{-\mu at-\mu A\mathcal{W}_t}]\times\exp\left[-s\mu b\int_0^t dz\, e^{-\mu az-\mu A\mathcal{W}_z}\right] \tag{3.1a}$$

or

$$x_t=\int_0^\infty ds\frac{s^{\nu-1}}{\Gamma(\nu)}e^{-sx_0^{-\mu}}e^{at+A\mathcal{W}_t}\times\exp\left[-s\mu b\int_0^t dz\, e^{\mu az+\mu A\mathcal{W}_z}\right]\ , \tag{3.1b}$$

where (for simplicity) $\nu>0$ $(\nu=1/\mu)$ is assumed.

Equation (3.1a) results from the change of variable of integration $(s\rightarrow t-s)$ in (2.2) and from stationarity of increments of $\mathcal{W}$. Expanding the exponents in (3.1a) we get

$$\langle x_t \rangle = \int_0^\infty ds \frac{s^{\nu-1}}{\Gamma(\nu)} \sum_{j=0}^\infty \frac{(-s)^j}{j! x_0^{j\mu}} \sum_{k=0}^\infty (-s\mu b)^k \Phi_k^{\mathcal{W}}(t; -j\mu a, -j\mu A, -\mu a, -\mu A) , \tag{3.2}$$

where

$$\begin{aligned}
\Phi_k^{\mathcal{W}}(t;P,R,p,r) &= \left\langle \frac{e^{Pt} e^{R\mathcal{W}_t}}{k!} \left[\int_0^t dz\, e^{pz} e^{r\mathcal{W}_z} \right]^k \right\rangle \\
&= \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k \exp[P(t-s_1)] \exp[(P+p)(s_1-s_2)] \cdots \exp[(P+kp)s_k] \\
&\quad \times \langle \exp(R\mathcal{W}_{t-s_1}) \rangle \langle \exp[(R+r)\mathcal{W}_{s_1-s_2}] \rangle \cdots \langle \exp[(R+kr)\mathcal{W}_{s_k}] \rangle \\
&= \exp\{[P+\lambda\varphi(R)]t\} * \cdots * \exp\{[P+kp+\lambda\varphi(R+kr)]t\} \\
&= \mathcal{L}_z^{-1} \left[\prod_{i=0}^k [z-(P+ip)-\lambda\varphi(R+ir)]^{-1} \right] \\
&= \sum_{i=0}^k \frac{\exp\{[P+ip+\lambda\varphi(R+ir)]t\}}{\prod_{\substack{m=0\\(m\neq i)}}^k \{(i-m)p+\lambda[\varphi(R+ir)-\varphi(R+mr)]\}} ,
\end{aligned} \tag{3.3}$$

where the asterisks indicate convolution of functions. The first equality of (3.3) defines the function $\Phi_k^{\mathcal{W}}$. In the second, the stationarity and independence of increments of $\mathcal{W}$ is applied to factorize the averaging. Next, using Eq. (2.5) we obtain $\Phi_k^{\mathcal{W}}$ in the form of convolution, evaluated finally as the inverse Laplace transformation (it was assumed, to simplify the notation, that all poles are simple) of a product of appropriate Laplace transforms.

Introducing the result of (3.3) into the right-hand side of Eq. (3.2) we obtain an expression containing the triple sum, which, after some reordering, takes the form

$$\langle x_t \rangle = \int_0^\infty ds \frac{s^{\nu-1}}{\Gamma(\nu)} \sum_{j=0}^\infty (-s)^j F_\pm(s,j) f_\pm(j,t) , \tag{3.4}$$

where

$$\begin{aligned}
F_-(s,j) &= \sum_{k=0}^\infty (-s\mu b)^k \Big/ \prod_{n=1}^k [\Omega_-(j)-\Omega_-(j+n)] , \\
f_-(j,t) &= \frac{e^{\Omega_-(j)t}}{x_0^{j\mu}} \sum_{i=0}^j \left[\frac{(\mu b x_0^\mu)^i}{(j-i)!} \Big/ \prod_{n=1}^i [\Omega_-(j)-\Omega_-(j-n)] \right] , \\
\Omega_-(\kappa) &= -\kappa\mu a + \lambda\varphi(-\kappa\mu A) ,
\end{aligned} \tag{3.5a}$$

and where function φ is related to white noise ξ_t via Eqs. (2.5) and (2.3). A similar procedure applied to Eq. (3.1b) leads to the representation (3.4) with

$$\begin{aligned}
F_+(s,j) &= e^{-sx_0^{-\mu}} \sum_{k=0}^\infty (s\mu b)^k \Big/ \prod_{n=1}^k [\Omega_+(j+n)-\Omega_+(j)] , \\
f_+(j,t) &= e^{\Omega_+(j)t} (\mu b)^j \Big/ \prod_{n=1}^j [\Omega_+(j)-\Omega_+(j-n)] , \\
\Omega_+(\kappa) &= \Omega_-(-\nu-\kappa) = a + \kappa\mu a + \lambda\varphi(A+\kappa\mu A) ,
\end{aligned} \tag{3.5b}$$

which, although formally equivalent to the former, may in some cases be more suitable.

Equation (3.4) and the scaling rule (2.4) give the formal solution of the kinetics (1.2), (2.1) driven by (any) white noise ξ_t. However, the practical usefulness of these formulae is limited for two reasons: (i) The term-by-term integration in (3.4) frequently cannot be done [even in the case of (3.5b)], and therefore the detailed knowledge of the behavior of $F(s,j)$ is required. (ii) The series (3.4) may contain an infinite number of terms (of different signs) which become divergent, when t grows to infinity, and thus the reordering and resummation is required in order to obtain the true time dependence in the whole domain of parameters. Unfortunately, as long as the products in the denominators of (3.5a) or (3.5b) are not *explicitly* expressed as some "simple" functions (with respect to j and k), it is difficult to say anything about properties of the series.

Nevertheless, there exists a class of functions φ, corresponding to some specific noises ξ_t, for which our F and

f turn out to be the series representation of some *special* functions. In particular, for φ given by Eq. (2.6) or (2.9) (corresponding to GWN or WSN with exponentially distributed weights, respectively), and more generally, when $\varphi(y)$ is a rational function [19] of y, it is possible to express products in Eq. (3.5a) and (3.5b) in terms of Pochhammer symbols $\{u\}_m = u(u+1)\cdots(u+m-1) = \Gamma(u+m)/\Gamma(u)$. Then functions $F(s,j)$ become the appropriate (generalized) hypergeometric functions of well-known properties [20].

In the case of GWN, we get from Eqs. (3.4) and (2.6)

$$F_-(s,j) = {}_0F_1(;1+2j-h;sg) ,$$
$$f_-(j,t) = e^{D\mu^2 j(j-h)t}\frac{x_0^{-j\mu}}{j!}\,{}_1F_1(-j;1-2j+h;gx_0^{\mu}) , \tag{3.6a}$$

or

$$F_+(s,j) = e^{-sx_0^{-\mu}}\,{}_0F_1(;h+2\nu+2j+1;sg) ,$$
$$f_+(j,t) = e^{D\mu^2(j+\nu)(j+\nu+h)t}\frac{g^j\Gamma(h+2\nu+j)}{j!\Gamma(h+2\nu+2j)} , \tag{3.6b}$$

where $h=a/D\mu$, $g=b/D\mu$, and $D=\lambda A^2$. For WSN with exponentially distributed weights, introducing rescaled variables

$$\bar a = a/\lambda,\quad \bar b = b/\lambda,\quad \bar t = \lambda t,\quad \bar A = A/\beta \tag{3.7}$$

(and omitting bars, to simplify the notation) and using (3.4) and (2.9), we obtain

$$F_-(s,j) = {}_1F_1(j+\nu/A+1;j+\vartheta_-(j)+1;-sQ) ,$$
$$f_-(j,t) = e^{\eta_-(j)t}\frac{(-Q)^j}{j!}\frac{\Gamma(\nu/A+j)\Gamma(\vartheta_-(j))}{\Gamma(\nu/A)\Gamma(\vartheta_-(j)+j)}$$
$$\times {}_2F_1(-j,\vartheta_-(j);\nu/A;x_0^{-\mu}/Q) , \tag{3.7a}$$

or

$$F_+(s,j) = e^{-sx_0^{-\mu}}\,{}_1F_1\left[j+\nu-\frac{\nu}{A}+1;j+\vartheta_+(j)+1;sQ\right] ,$$
$$f_+(j,t) = e^{\eta_+(j)t}\frac{Q^j}{j!}\frac{\Gamma(\nu-\nu/A+j)\Gamma(\vartheta_+(j))}{\Gamma(\nu-\nu/A)\Gamma(\vartheta_+(j)+j)} , \tag{3.7b}$$

where $Q=b/(a-A)$ and

$$\vartheta_-(z) = \frac{\nu}{A} + \frac{\nu}{(a-A)(1+z\mu A)} ,$$
$$\vartheta_+(z) = \nu - \frac{\nu}{A} - \frac{\nu}{(a-A)(1-A-z\mu A)} ,$$
$$\eta_-(z) = -\mu(a-A)z - 1 + \frac{1}{1+z\mu A} ,$$
$$\eta_+(z) = (a-A)(1+\mu z) - 1 + \frac{1}{1-A-z\mu A} .$$

Note that, in contrary to the case of GWN, the sign of the parameter A (to which the noise is coupled) plays a significant role, which is related to an intrinsic asymmetry of the noise. The analysis of properties of $\langle x_t\rangle$ in different regions of parameters space is given in the next section.

IV. TIME-DEPENDENT SOLUTIONS

We begin this section with a short presentation of useful technique of handling the alternating power series, which is known as the Mellin-Barnes representation (sometimes referred to as the Borel-summation method). Suppose that we have a power series of complex variable z:

$$\sum_{k=0}^{\infty} C(k)\frac{(-z)^k}{k!} , \tag{4.1}$$

where $C(s)$ is analytical in a right half-plane $\mathrm{Re}(s)\geq 0$, except, maybe, at a finite number of points, where it has the poles. Suppose next that we want to know the analytical continuation of (4.1) for some values of z outside of the disk of convergence. Consider the contour integral

$$\frac{1}{2\pi i}\int ds\,\Gamma(-s+\sigma)C(s-\sigma)z^{-\sigma+s} , \quad |\arg(z)|<\pi , \tag{4.2}$$

where the path of integration starts and ends in infinity, and separates the poles of $\Gamma(-s+\sigma)$ from those of $C(s-\sigma)$. σ is an auxiliary parameter usually taken as zero. This integral in regions of z, where it is for the appropriate choice of path convergent [21], usually gives the representation we are looking for. In fact, for z belonging to the common area of convergence, the evaluation of (4.2) as a (minus) sum of residues on the right of the path leads just to Eq. (4.1).

A. The case of Gaussian white noise

This case has been examined by the use of several methods [22]. The final result may be easily obtained from Eqs. (3.4) and (3.6b) because the term-by-term integration is possible in this case. The resulting series requires analytical continuation in the long-time limit. It has been done, using Mellin-Barnes (MB) representation, in Refs. [23,18].

New technique details, which will be especially important in Sec. IV B appear when one wants to recover the results (cf., e.g., [18]) starting from (3.4) and (3.6a). Such an analysis is presented in Appendix A.

The asymptotic (stationary) behavior of $\langle x_t\rangle$ depends on values of parameters a,b,μ,D in the following way [18]:

$$\langle x\rangle_{\mathrm{st}} = \begin{cases} \dfrac{\Gamma(h+\nu)}{g^{\nu}\Gamma(h)} & \text{if } a,\mu>0 \text{ or } a<-D<0<-\mu \\ 0 & \text{if } \mu>0\ ,\ a<0 \\ \infty & \text{if } \mu<0\ ,\ -D\leq a \end{cases} \tag{4.3}$$

where $\langle x\rangle_{\mathrm{st}} = \lim_{t\to\infty}\langle x_t\rangle$.

B. The case of white shot noise (with exponentially distributed weights)

Introducing (3.7a) or (3.7b) into Eq. (3.4) we get

$$\langle x_t\rangle=\int_0^\infty ds\frac{s^{\nu-1}}{\Gamma(\nu)}\sum_{j=0}^{\infty}\frac{(\pm sQ)^j}{j!}\frac{\Gamma(\vartheta_\mp(j))\Gamma(j+v_\mp)}{\Gamma(j+\vartheta_\mp(j))\Gamma(v_\mp)}e^{\eta_\mp(j)t}\,{}_1F_1(j+v_\mp+1;j+\vartheta_\mp(j)+1;\mp sQ)G_\mp(s,j)\ , \tag{4.4}$$

where $v_-=\nu/A$ and $G_-(s,j)={}_2F_1(-j,\vartheta_-(j);\nu/A;x_0^{-\mu}/Q)$, or $v_+=\nu-\nu/A$ and $G_+(s,j)=e^{-sx_0^{-\mu}}$, respectively. The way of handling Eq. (4.4), which expresses the mean value as a function of rescaled parameters (3.7), depends essentially on the signs of two composite parameters $\mu(a-A)$ and $(a-A)A$. The former determines the sign of $Q=b/(a-A)$ (remember that we are working with a condition $b\mu>0$), the latter specifies the qualitative character of the behavior of $\eta_\mp$. Namely, for positive $(a-A)A$, $\eta_\mp$ is monotonic both to the left and right of its pole (equal to $-v_\mp$), whereas it has a local minimum on the one and a local maximum on the other side of its vertical asymptote, for $(a-A)A<0$. In the first case, the zeros of $\eta_\mp(z)$ are intersected by the pole, and in the opposite case, they are separated by the local minimum and lie on the same side of the vertical asymptote.

There are four different cases with respect to the signs of the above-mentioned parameters:

$A(a-A)$ \ $\mu(a-A)$	+	−
+	$0<\mu$, $0<A<a$ $\mu<0$, $a<A<0$ case I (upper)	$0<\mu$, $a<A<0$ $\mu<0$, $0<A<a$ case II (lower)
−	$A<0<\mu$, $A<a$ $\mu<0<A$, $a<A$ case IV(MB,lower)	$0<\mu$, A , $a<A$ μ , $A<0$, $A<a$ case III (MB,upper)

In cases I and III, Eq. (4.4) with upper signs will be used. For cases II and IV the lower signs in (4.4) will be chosen. Cases III and IV require the reexpression of the series in order to carry out the integration in (4.4).

Case I corresponds, e.g., to the case when, say, a small positive parameter of noise coupling (A) does not exceed a deterministic one (a), and $\mu>0$. Choosing the upper signs in Eq. (4.4) and provided $1+\nu/A-\nu>0$, one can termwise integrates in (4.4), using the formula [20]

$$\int_0^\infty dy\, y^{\alpha-1}\,{}_1F_1(\gamma;\delta;-xy)=\frac{\Gamma(\alpha)\Gamma(\delta)\Gamma(\gamma-\alpha)}{x^\alpha\Gamma(\gamma)\Gamma(\delta-\alpha)}\ ,$$

$$\mathrm{Re}(\alpha),\mathrm{Re}(\gamma-\alpha),\mathrm{Re}(x)>0\ ,$$

and obtaining

$$\langle x_t\rangle=\frac{\Gamma\left[1+\frac{\nu}{A}-\nu\right]}{Q^\nu\Gamma(\nu/A)}\times\sum_{j=0}^{\infty}\frac{\{\nu\}_j\Gamma(\vartheta_-(j))[j+\vartheta_-(j)]}{j!\Gamma(1-\nu+\vartheta_-(j))[j+\nu/A]}\times\exp[\eta_-(j)t]\,{}_2F_1\left[-j;\vartheta_-(j);\frac{\nu}{A};\frac{x_0^{-\mu}}{Q}\right]. \tag{4.5}$$

All terms of the series (4.5), except the first one, decay exponentially (with rate growing with j) with time, so the stationary mean value reads

$$\langle x\rangle_{\mathrm{st}}=Q^{-\nu}\frac{\Gamma\left[1-\nu+\frac{\nu}{A}\right]\Gamma\left[1+\frac{\nu a}{A(a-A)}\right]}{\Gamma\left[1-\nu+\frac{\nu a}{A(a-A)}\right]\Gamma\left[1+\frac{\nu}{A}\right]}\ ,$$

$$1-\nu+\nu/A>0\ , \tag{4.6}$$

and a relaxation is governed by the second term of the series.

Case II may be examined in a similar manner. Carrying out the integration in (4.4) with lower signs, one obtains

$$\langle x_t\rangle=x_0\sum_{i=0}^{\infty}\frac{(-Qx_0^\mu)^i\{\nu\}_i\{\nu-\nu/A\}_i}{i!\{\vartheta_+(i)\}_i}\exp[\eta_+(i)t]\times{}_2F_1\left(\nu+i,1+\nu-\frac{\nu}{A}+i;1+i+\vartheta_+(i);Qx_0^\mu\right)\ . \tag{4.7}$$

Here, and hereafter, the Gauss hypergeometric function of a variable $y\le-1$ is understood as given by the analytical continuation (5.1).

If $\mu>0$, all terms relax to zero ($\langle x\rangle_{\mathrm{st}}=0$), and it is not unexpected because then both linear and nonlinear parts of Eq. (1.2) describe annihilation. If $\mu<0$ and $0<A<1$, $\eta_+(i)$ is positive for $0\le i<-\nu$, and $\langle x_t\rangle$ growths infinitely. For $\mu<0$ and $A>1$, the mean value of x_t does not exist for any $t>0$ (and for any a). It does not result from character of evolution, but it is a trivial consequence of the fact that the average over (exponentially distributed) weights ξ does not exist in this case, which is clearly seen from Eqs. (2.2), (2.8), and (2.9). Therefore the case of $\mu<0$ and $A>1$ should be excluded from all considerations.

Cases III and IV will be examined, by the use of Mellin-Barnes method. Equation (4.4) with upper signs in the former and with lower signs in the latter case should be chosen. Then, the (alternating) series in Eq. (4.4) may be replaced by the Mellin-Barnes integral (4.2) with $\sigma=v$:

$$\vartheta=\int \frac{dz}{2\pi i}\frac{\Gamma(v-z)|sQ|^{z-v}\Gamma\left[v-\frac{R^2}{z}\right]\Gamma(z)}{\Gamma\left[z-\frac{R^2}{z}\right]\Gamma(v)}\exp[\eta(z-v)t]\,{}_1F_1\left[z+1;z-\frac{R^2}{z}+1;|sQ|\right]\left\{\begin{array}{l}{}_2F_1\left[v-z;v-\frac{R^2}{z};v;\frac{x_0^{-\mu}}{Q}\right]\\ e^{-sx_0^{-\mu}}\end{array}\right. , \tag{4.8}$$

where $R=|\nu|/\sqrt{(A-a)A}$ and $v=v_{\mp}$, $\eta=\eta_{\mp}$, respectively. The path of integration should separate the poles $v_k=v+k$ of $\Gamma(v-z)$ from those of $\Gamma(v-R^2/z)$ [i.e., from poles at points $\tilde{v}_k$ being images of v_k by the inversion with respect to the circle $C(0,R)$], and those of $\Gamma(z)$ ($m_k=-k$), $k=0,1,2,\ldots$. The point $z=0$ is the *essential singularity* of the integrand, and lies to the left of all v_k.

In order to carry out the integration over s, the reexpression of ϑ is still required. This important, but very technical, part of the analysis is presented in Appendix B. The final result reads

$$\langle x_t\rangle=\tfrac{1}{2}\oint\frac{dz}{2\pi i}W(z)-\sum_{k=0}^{[R-v]}r(v+k)+{\sum_{k=0}^{\infty}}' r(v-\nu-k)+\sum_{k=[R]}^{\infty}r(-1-k) , \tag{4.9}$$

where

$$W(z)=|Q|^{-\nu}e^{[\eta(z-v)t]}\frac{\Gamma(-z+v)\Gamma\left[v-\frac{R^2}{z}\right]\Gamma(\nu-v+z)\Gamma\left[\nu-v+\frac{R^2}{z}\right]}{\Gamma(\nu)\Gamma(v)\Gamma\left[\frac{R^2}{z}-z\right]\Gamma\left[z-\frac{R^2}{z}\right]}G(z) ,$$

and $r(u)=\mathrm{res}[W(z);z=u]$, $[w]=\mathrm{entire}(w)$. The function $G(z)$ is defined below Eq. (B1) in Appendix B. The integral is taken along the counterclockwise-oriented circle $C(0,R)$, and the prime indicates that terms with $k\in[v-\nu-R, v-\nu+R]$ (i.e., the appropriate residues at points lying inside the circle) in the middle sum should be removed. The first sum, if empty, is treated as zero; the last does not appear in case III.

The integral (which evaluates a contribution from the continuous spectrum) in the angular parametrization takes the form which is *explicitly* real

$$X_{\mathrm{sc}}(t)=\int_{-\pi}^{\pi}d\phi\frac{R\sin\phi\sinh(2\pi R\sin\phi)\left|\Gamma\left[\frac{\nu}{A}-\xi\right]\Gamma\left[\nu-\frac{\nu}{A}+\xi\right]\right|^2}{2\pi^2|Q|^{\nu}\Gamma(\nu)}e^{(a/A-2+2\sqrt{1-a/A}\cos\phi)t}Y(\phi) , \tag{4.9a}$$

where either $\xi=\mathrm{Re}^{i\phi}$ and

$$Y(\phi)=\frac{\Gamma\left[1-\nu+\frac{\nu}{A}\right]}{\Gamma\left[\frac{\nu}{A}\right]}{}_2F_1\left[\frac{\nu}{A}-\xi;\frac{\nu}{A}-\xi^*;\frac{\nu}{A};-\frac{x_0^{-\mu}}{|Q|}\right] ,$$

or $\xi=-\mathrm{Re}^{i\phi}$ and

$$Y(\phi)=\frac{|\Gamma(1+\mathrm{Re}^{i\phi})|^2}{\Gamma\left[\nu-\frac{\nu}{A}\right]\Gamma\left[1+\frac{\nu}{A}+2R\cos\phi\right]}\times{}_2F_1\left[\frac{\nu}{A}-\xi;\frac{\nu}{A}-\xi^*;\frac{\nu}{A}+1+2R\cos\phi;1-\frac{x_0^{-\mu}}{|Q|}\right]$$

in cases III and IV, respectively. The residues $r(v+\kappa)$, which appear in the first and second sum, may be commonly written as

$$r(v_{\mp}+\kappa)=\frac{|Q|^{-\nu}\Gamma(\nu+k)[\kappa+\vartheta_{\mp}(\kappa)]}{k!\Gamma(\nu)\Gamma(v_{\mp})[\kappa+v_{\mp}]}H(\vartheta_{\mp}(\kappa))\times\exp[\eta_{\mp}(\kappa)t]Z_{\mp}(\kappa) , \tag{4.9b}$$

where

$$H(y)=\Gamma(\nu-y)/\Gamma(1-y)\quad\text{if } \kappa=k$$

or

$$H(y)=\Gamma(y)/\Gamma(1-\nu+y)\quad\text{if } \kappa=-\nu-k$$

and where

$$Z_{-}(\kappa)=\Gamma\left[1-\nu+\frac{\nu}{A}\right]{}_2F_1\left[-\kappa;\vartheta_{-}(\kappa);\frac{\nu}{A};-\frac{x_0^{-\mu}}{|Q|}\right]$$

or

$$Z_+(\kappa)=\frac{\Gamma(1+v_+ +\kappa)\Gamma(1+v_+ -\vartheta_+(\kappa))}{\Gamma(1+\nu+v_+ +\kappa-\vartheta_+(\kappa))}\times{}_2F_1\left[\nu+\kappa;\nu-\vartheta_+(\kappa);1+\nu+v_+ +\kappa-\vartheta_+(\kappa);1-\frac{x_0^{-\mu}}{|Q|}\right].$$

The last sum appears only in case IV. Provided ν/A is not a negative integer, its summands may be expressed as

$$r(-j)=\frac{\Gamma\left[\frac{\nu}{A}-1\right]\Gamma\left[\nu-\frac{\nu}{A}+j\right]\Gamma\left[\nu-\frac{\nu}{A}+\frac{R^2}{j}\right]\left[1-\frac{R^2}{j^2}\right]}{x_0^{-1}(j-1)!|Qx_0^{\mu}|^{1+\nu-\nu/A}\Gamma(\nu)\Gamma\left[\nu-\frac{\nu}{A}\right]\Gamma\left[\frac{R^2}{j}\right]}\times e^{[a/A+\nu/jA-2-j\mu(a-A)]t}{}_2F_1\left[1-j;1-\frac{R^2}{j};2-\frac{\nu}{A};\frac{x_0^{-\mu}}{|Q|}\right]. \tag{4.9c}$$

Equations (4.9) solve the kinetics in regions of parameters space covered by cases III and IV, giving the spectral decomposition of moments of x_t. In contrast to (4.5) and (4.7) the continuous spectrum appears. It fills up the gap between values taken by η on the left and on the right of its vertical asymptote. $X_{sc}(t)\to 0$, if t grows to infinity. Only finite number of terms, which do not decay exponentially in time, may appear, namely as the first few terms of the left or middle sum in Eq. (4.9). Elementary analysis, which is not presented here, shows that, for the positive ν, $\langle x_t\rangle$ either approaches (finite) stationary value if $a>0$, or relaxes to zero, otherwise (see Fig. 1). For negative ν and A, the mean value growths infinitely if $a>A^2/(A-1)$, or remains finite for $a\in[A,A^2/(A-1)]$. And, for $\nu<0$ and positive A, $\langle x_t\rangle\to\infty$ if a is positive, or remains finite, otherwise. (Remember that, in the latter case, we consider $A<1$ only.)

The stationary value (if positive finite) is given by (4.9b) with $k=0$ and reads

$$\langle x\rangle_{st}=|Q|^{-\nu}\frac{\Gamma\left[1-\nu+\frac{\nu}{A}\right]\Gamma\left[\nu+\frac{\nu a}{A(A-a)}\right]}{\Gamma\left[1+\frac{\nu}{A}\right]\Gamma\left[\frac{\nu a}{A(A-a)}\right]}, \tag{4.10}$$

in case III, or

$$\langle x\rangle_{st}=Q^{-\nu}\frac{\Gamma\left[-\frac{\nu}{A}\right]\Gamma\left[\nu+\frac{\nu a}{A(A-a)}\right]}{\Gamma\left[\nu-\frac{\nu}{A}\right]\Gamma\left[\frac{\nu a}{A(A-a)}\right]},$$

$$\nu-\nu/A>0\;, \tag{4.11}$$

in case IV. The asymptotic behavior in different areas of parameter space is shown in Fig. 1.

V. NUMERICAL RESULTS

In order to confirm the analytical results of Sec. IV B, the digital averaging of (2.2) has been done. To this end, for an arbitrary given ending time T, the particular realization of (standardized) compound Poisson process $\mathcal{W}$ has to be chosen. It is the piecewise constant function, having T_* steps at some points t_i uniformly distributed on $(0,T)$, each one of a random height ζ_i governed by exponential distribution $p(\zeta)=e^{-\zeta}$. T_* is a random integer, chosen according to Poisson distribution $P(T_*=N)=e^{-T}T^N/N!$. For each realization $\{t_i,\zeta_i: i=0,1,\ldots\}$ $(t_0=0,\zeta_0=0)$ the value of x_t is given elementary by

$$x_t(\{t_i,\zeta_i\})=x_0e^{(a-A)t}e^{A\sigma_{i(t)}}\times\left[1+\mu bx_0^{\mu}\times\sum_i^{i(t)}e^{\mu A\sigma_i}\int_{t_i}^{t_{i+1}}ds\,e^{\mu(a-A)s}\right]^{-1/\mu},$$

where $\sigma_0=\zeta_0$, $\sigma_{k+1}=\sigma_k+\zeta_{k+1}$, and $t_{i(t)+1}=t\le T$. The $\langle x_t\rangle$ is then calculated as the arithmetical average of several thousands, obtained for different sample realizations, values.

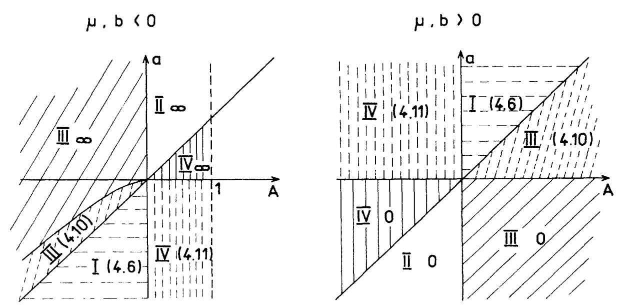

FIG. 1. Asymptotic behavior of $\langle x_t\rangle$ in different regions of the (A,a) plane. The left and right graphs correspond to $b,\mu<0$ and $b,\mu>0$, respectively. The labels of appropriate cases and corresponding stationary values $\langle x\rangle_{st}$ are listed. The condition of validity of (4.6) and (4.10), $1-\nu+\nu/A>0$, involves the particular value of ν, and therefore it is not shown. The curve intersecting area III on the left graph is given by $a=A^2/(A-1)$.

On the other hand, the (analytical) mean value is computed by truncating the infinite series in Eq. (4.5), (4.7), or (4.9), respectively (after approaching the required level of accuracy). Their radii of convergence turn out to be given by $r<1$,

$$r \propto x_0^{\alpha} e^{-\Lambda t} ,$$

where $\Lambda > 0$ and α are some parameters. Therefore these series are convergent for any arbitrary x_0, at least for sufficiently large t. The Gauss hypergeometric function (if it is not simply a polynomial) is computed the same way. At least one of its equivalent (in the analytical continuation sense) series representations

$$_2F_1(a;b;c;y)=(1-y)^{-a}\,_2F_1\left[a;c-b;c;\frac{y}{y-1}\right] \quad (5.1)$$

is convergent, and may be directly used.

Finally, the integral (4.9a) has been numerically evaluated, using Romberg technique. The comparison is presented in Figs. 2–5. The agreement is very good for all times. The interesting effect is observed for Eq. (1.2) describing creation and annihilation processes (i.e., for a,b,μ having the same sign) if the initial state is close to the finite stationary value. Namely, as it is shown in Figs. 2–4, at the beginning of evolution, the mean value increases and approaches the local maximum. In contrast, the deterministic relaxation is always monotonic.

This type of behavior may be explained as follows: From Eq. (3.3) one can easily obtain (e.g., using Laplace transforms) that

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_k^{\mathcal{W}}(t;P,R,p,r)=\begin{cases} P+\lambda\varphi(R) & \text{if } k=0 \\ 1 & \text{if } k=1 \\ 0 & \text{if } k=2,3,\ldots . \end{cases}$$

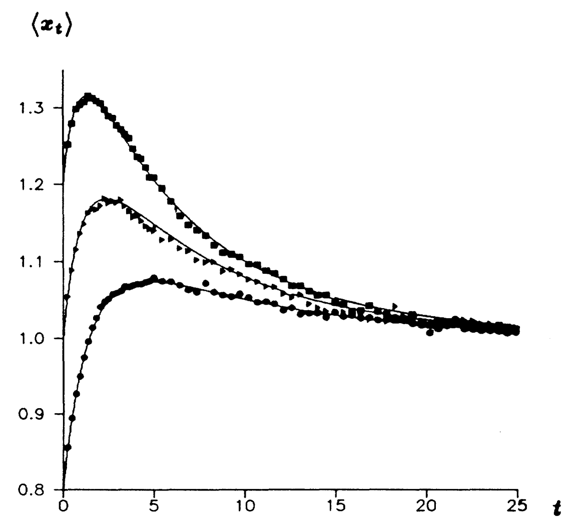

FIG. 2. Comparison of analytical results and digital simulation. Plots $\langle x_t \rangle$ vs t (rescaled, dimensionless time) for different initial values $x_0=0.8, 1.0, 1.2$; and $\nu=1$ (true Verhulst model), $a=0.2$, $b=0.2$, $A=0.4$ (case III). Each mark represents the arithmetical average over $N=50\,000$ values obtained for different sample realizations.

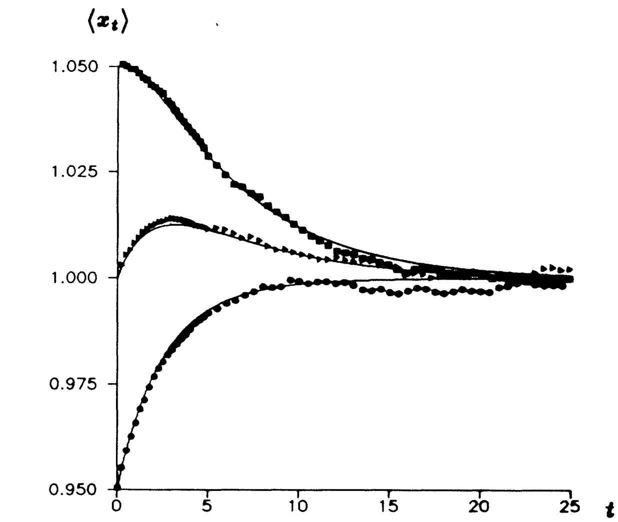

FIG. 3. Plots $\langle x_t \rangle$ vs t for $x_0=0.95$, 1.0, 1.05; and $\nu=1$, $a=0.2$, $b=0.2$, $A=0.1$ (case I). $N=50\,000$.

It gives ($A<1$ [24])

$$\langle \dot{x}_{t=0^+} \rangle = x_0[a+\varphi(A)-bx_0^{\mu}] .$$

The sign of the last expression determines the character of the evolution at the beginning. It may act against the global trend $x_0 \rightarrow \langle x \rangle_{\rm st}$.

Consider, for instance, the true Verhulst model ($\nu=1$), for which the stationary mean value is not affected by the noise, being equal to the deterministic stationary state $x_{\rm st}=(b/a)^{-1/\mu}$. [This follows immediately from (4.3) for GWN, and from appropriate equations in Sec. IV B for WSN, and seems to be some general rule.] Therefore, for all x_0 from $[x_{\rm st}, x_{\rm st}+\varphi(A)/b]$, the effect described above will be observed [25]. For negative $a,b,\mu<-1$ the reverse effect may appear, viz. $\langle x_t \rangle$ may have a local minimum; see Fig. 5.

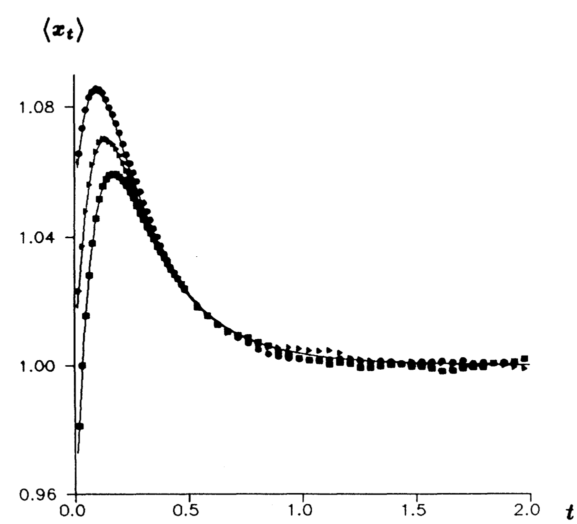

FIG. 4. Plots $\langle x_t \rangle$ vs t for $x_0=0.95$, 1.0, 1.05; and $\nu=1$, $a=10$, $b=10$, $A=-1$ (case IV). $N=100\,000$.

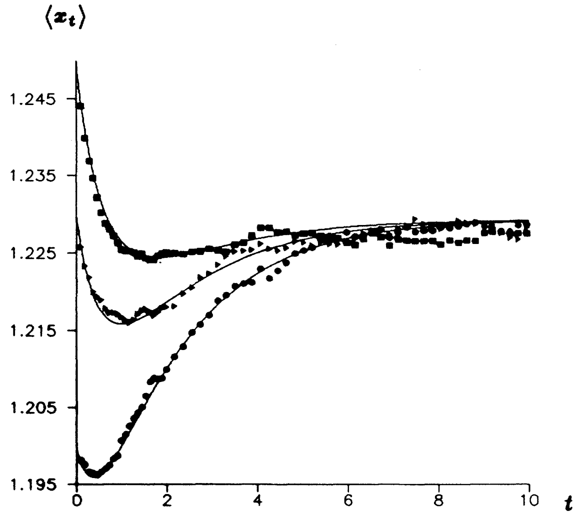

FIG. 5. Plots $\langle x_t \rangle$ vs t for $x_0=1.20$, 1.23, 1.25; and $\nu=-0.5$, $a=-0.45$, $b=-0.45$, $A=-0.4$ ($N=100\,000$).

VI. FINAL REMARKS

This paper has been devoted to the analytical studying of the evolution of the Verhulst-type model in the presence of linearly coupled white noise. Our main results are (i) the general formula (3.4) for (transient) moments of x_t, which is valid for any type of white noise ξ_t present; and (ii) the exact time-dependent solution of kinetics driven by white shot noise (with exponentially distributed weights), contained in Eqs. (4.5), (4.7), and (4.9).

(i) Equation (3.4) gives, for any white noise ξ_t, the moments of the process x_t (with the initial condition $x_{t=0}=x_0$), in the form of the Mellin or Laplace transforms of some function represented formally by series expansion. The direct averaging method presented here does not in principle require that noise is white, and may also be applied in the case of colored noise. Similarly, the deterministic coefficients in Eq. (1.2) need not be constant. It gives several possibilities of consideration of more complicated models based on Bernoulli equation (1.2) (e.g., the cases of simultaneous presence of two noises, or of the presence of the noise and regular perturbation, etc. [26]). However, our simple general way (3.3) of calculating the appropriate $\Phi_k^{\mathcal{W}}$ is then not available, and therefore each case will require the special treating. [Moreover, if x_t is not a Markov process, it is also not completely characterized by its (one-point) transient moments.]

(ii) In Sec. IV the particular cases of GWN and WSN [27] have been analyzed by the use of contour (Mellin-Barnes) integral representation of appropriate series. Such procedure has enabled us to find the analytical continuation of the formal solution (3.4) in the whole domain of parameters. The final results, Eqs. (4.5), (4.7), and (4.9), give the spectral decomposition of transient moments $m_t(\omega;x_0)=\langle x_t^{\omega}\rangle$ for different interrelations between parameters. The asymptotic behavior ($t\to\infty$) has been examined (Fig. 1), and stationary mean values have been found (4.6), (4.10), and (4.11).

Several other quantities of interest may be easily obtained from the spectral decomposition of moments. Namely, the transition probability density is given by the inverse Mellin transformation

$$P(x,t|x_0)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, x^{-s-1} m_t(s;x_0)\ . \qquad (6.1)$$

If a,b,μ have the same sign, the stationary probability distribution [28] is found [$m_t(s;x_0)\equiv\langle x^s\rangle_{\rm st}$ in (6.1)] in the form

$$P_{\rm st}(x)\propto \chi(x) x^{-1-a/(a-A)A}|x^{\mu}-Q^{-1}|^{-1+\nu/(a-A)}\ , \qquad (6.2)$$

where

$$\chi(x)=\begin{cases}\Theta(Q^{-\nu}-x) & \text{if } A<0<a, \text{ or } a<A<0\\ \Theta(x-Q^{-\nu}) & \text{if } 0<A<a, \text{ or } a<0<A\\ \Theta(x) & \text{if } 0<a<A, \text{ or } A<a<0\ ,\end{cases}$$

and Θ is the unit-step function. This form complies with the general one obtained in Ref. [2] from the appropriate Fokker-Planck-type equation.

The probability density distribution and its moments follow from integrating of (6.1) or $m_t(\omega,x_0)$ over $x_0\in(0,\infty)$ (with some initial distribution), respectively. The autocorrelation function is given by the double integral

$$\langle x_{t+\tau}x_t\rangle=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\omega\, m_t(\omega;x_0)\int_0^{\infty} dy\, y^{-\omega}\langle x_\tau(y)\rangle\ ,$$

and the stationary autocorrelation function simply by

$$K_{\rm st}(\tau)=\int_0^{\infty} dy\, P_{\rm st}(y) y \langle x_\tau(y)\rangle\ .$$

The behavior of the latter quantity, in the case of GWN, has been examined in [23]. The properties of the autocorrelation function, in the case of WSN, will be studied in a separate paper. The comparison of the digital simulation of (2.2) (with appropriate compound Poisson process $\mathcal{W}$) and the numerical calculation of (4.5), (4.7), or (4.9) has been done in Sec. V (Figs. 2–5).

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APPENDIX A: CALCULATION OF THE MEAN VALUE IN THE CASE OF GWN

The way of handling Eqs. (3.4) and (3.6a) is the following: Function F_- and the (confluent) hypergeometric polynomial in (3.6a) may be expressed by a modified Bessel function and Whittaker function [20], respectively. Then (3.4) gives

$$\langle x_t\rangle = \frac{e^{gx_0^\mu/2}g^{-1/2}}{(x_0^\mu)^{h/2+1/2}}\int_0^\infty ds\frac{s^{\nu-1+h/2}}{\Gamma(\nu)}\mathcal{I}(s,t)\ , \qquad (\text{A1})$$

where

$$\mathcal{I}(s,t) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(2j-h)\Gamma(j-h)I_{2j-h}(2\sqrt{sg}\,) \times e^{D\mu^2 j(j-h)t}W_{1/2+h/2,h/2-j}(gx_0^\mu)\ .$$

The series in Eq. (A1) cannot be termwise integrated, so its reexpression is needed to carry out the integration. The proper way is to replace it by the Mellin-Barnes integral [corresponding to (4.2) with $\sigma=-h/2$]

$$\mathcal{I}[I_{2z}] = \int_{-i\infty}^{+i\infty}\frac{dz}{2\pi i}\Gamma\left[-z-\frac{h}{2}\right]2z\Gamma\left[z-\frac{h}{2}\right]I_{2z}(2\sqrt{sg}\,) \times e^{D\mu^2(z^2-h^2/4)t}W_{1/2+h/2,-z}(gx_0^\mu)\ , \qquad (\text{A2})$$

where the path of integration separates the poles of $\Gamma(-z-h/2)$ from those of $\Gamma(z-h/2)$. If we at first assume that (i) $h<0$ the path may be taken simply as the imaginary axis.

Then, by changing of variable $z\to -z$ in (A2), we obtain [29] $\mathcal{I}[I_{2z}]=\mathcal{I}[-I_{-2z}]$. Taking the sum of these two integrals, we easily get $\mathcal{I}[I_{2z}(2\sqrt{sq}\,)] = \mathcal{I}[-\sin(2z\pi)K_{2z}(2\sqrt{sg}\,)/\pi]$, where K is a modified Bessel function of the second kind. The point is that, in (A1) with $\mathcal{I}$ given by the last expression, we can carry out the integration over s, using the well-known formula [20]

$$\int_0^\infty ds\, s^{\beta-1}K_{2z}(2\sqrt{sg}\,) = \frac{g^{-\beta}}{2}\Gamma(\beta+z)\Gamma(\beta-z)\ ,$$

$$\text{Re}(\beta\pm z)>0\ ,$$

provided (ii) $\nu+h/2>0$. It gives

$$\langle x_t\rangle = i\mathcal{N}(t)\int_{-i\infty}^{+i\infty}dz\, e^{D\mu^2 z^2 t}z\sin(2\pi z)W_{1/2+h/2,-z}(gx_0^\mu) \times\Gamma\left[-z-\frac{h}{2}\right]\Gamma\left[z-\frac{h}{2}\right] \times\Gamma\left[\nu+\frac{h}{2}+z\right]\Gamma\left[\nu+\frac{h}{2}-z\right]\ , \qquad (\text{A3})$$

where

$$\mathcal{N}(t) = \frac{g^{-\nu}e^{gx_0^\mu/2}e^{-a^2t/4D}}{2\pi^2(gx_0^\mu)^{h/2+1/2}\Gamma(\nu)}\ .$$

The result may be formulated as follows. The mean value of x_t is given by the contour integral (A3), where the path of integration have all poles of $\Gamma(-z-h/2)\Gamma(-z+\nu+h/2)$ on its right-hand side, and all poles of $\Gamma(z-h/2)\Gamma(z+\nu+h/2)$ on its left-hand side. This formulation enables us to remove the specific restrictions (i) and (ii). The *explicit* form of the spectral decomposition of $\langle x_t\rangle$ follows from the evaluation (according to Cauchy theorem) of (A3) along the appropriately indented (in a general case) line.

APPENDIX B: CALCULATION OF THE MEAN VALUE IN THE CASE OF WSN

Equation (4.4) with the contour integral (4.8) instead of the series is examined:

$$\langle x_t\rangle = \int_0^\infty ds\frac{s^{\nu-1}}{\Gamma(\nu)}\mathcal{J}[{}_1F_1(z+1;z-R^2/z+1;|sQ|)]\ .$$

Our problem is to find the proper way of interchanging the integrations over s and z. To this end three "auxiliary" conditions will be assumed. Nevertheless, the final results will be extended on the whole domain of parameters.

The first assumption, (i) $R<v$, enables us to choose the contour of integration in (4.8) (described below the mentioned expression) as consisting of the line $\text{Re}(z)=-R$ and counterclockwise oriented circle of radius R at the origin. Call the integral over the line $\mathcal{J}_l$, and that over the circle $\mathcal{J}_0$. Define two new integrals $\mathcal{J}'_l$ and $\mathcal{J}'_0$, where

$$\mathcal{J}' = \mathcal{J}\left[\frac{\Gamma(1+R^2/z)}{\Gamma(R^2/z-z)}\psi(z+1;z-R^2/z+1;|sQ|)\right]$$

and where

$$\psi(\alpha;\gamma;x) = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}{}_1F_1(\alpha;\gamma;x) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}x^{1-\gamma}{}_1F_1(1+\alpha-\gamma;2-\gamma;x)$$

is the second solution of Kummer equation.

The change of integration variable $z\to R^2/z$ in $\mathcal{J}_0$ leads just to the integral containing the second part of ψ, and thus

$$\mathcal{J}_0 = \tfrac{1}{2}\mathcal{J}'_0\ .$$

Moreover, under the condition (ii) $R<1$, we have also

$$\mathcal{J}_l = \mathcal{J}'_l\ ,$$

because then both integrands have the same poles, with the same residues, on the left-hand side of l. Namely, they are the residues at negative integers z, for which the second parts of appropriate $\psi(-k;\gamma;x)$ vanish.

Introducing $\mathcal{J}'_l+\frac{1}{2}\mathcal{J}'_0$ instead of the series into Eq. (4.4) and assuming (iii) $\nu>R+v$, we may interchange the order of integration, and carry out the integration over s by the use of formula [20];

$$\int_0^\infty d\xi\,\xi^{b-1}\psi(a;c;q\xi)e^{-y\xi} = \frac{\Gamma(b)\Gamma(b-c+1)}{q^b\Gamma(a+b-c+1)} \times{}_2F_1\left[b;b-c+1;a+b-c+1;1-\frac{y}{q}\right] \xrightarrow[y\to 0]{} \frac{\Gamma(b)\Gamma(b-c+1)\Gamma(a-b)}{q^b\Gamma(a)\Gamma(a-c+1)}\ ,$$

where $\text{Re}(b),\text{Re}(b-c+1),q,y>0$. The limiting form (for $y=0$) is valid if, moreover, $\text{Re}(a-b)>0$. It gives

$$\langle x_t\rangle=\frac{|Q|^{-\nu}}{2\pi i}\left[\int_l+\frac{1}{2}\oint\right]dz\, e^{\eta(z-v)t}\frac{\Gamma(-z+v)\Gamma(v-R^2/z)\Gamma(\nu-v+z)\Gamma(\nu-v+R^2/z)}{\Gamma(\nu)\Gamma(v)\Gamma(R^2/z-z)\Gamma(z-R^2/z)}G(z)\ , \tag{B1}$$

where

$$G(z)=z^{-1}\Gamma(1-\nu+v)$$
$$\times{}_2F_1(v-z;v-R^2/z;v;-x_0^{-\mu}/|Q|)$$

in case III (if, moreover, $1-\nu+v>0$), and

$$G(z)=\frac{\Gamma(1+z)\Gamma(1+R^2/z)}{z\Gamma(\nu-v+z+R^2/z+1)}$$
$$\times{}_2F_1\left[\nu-v+z;\nu-v+R^2/z;\nu-v+z+R^2/z+1;1-\frac{x_0^{-\mu}}{|Q|}\right]$$

in case IV.

Now we want to remove the specific restrictions (i), (ii), and (iii), which are not essential for the convergence of the integrals in (B1). For this purpose, note that the integrand of (B1) has the following location of poles: all the poles $v_k=v+k$ and $w_k=v-\nu-k$ (and $m_1=-1, m_2=-2,\dots,$ in case IV) lie outside the circle [because of (i), (iii), and (ii), respectively]. Therefore all the remaining poles $\tilde{v}_k$ and $\tilde{w}_k$ and (and $\tilde{m}_k$ in case IV) are inside, as the images of the former poles by the inversion with respect to this circle. Only the poles w_k (and m_k) lie on the left-hand side of l, and the first integral in (B1) may be evaluated as the sum of residues at these poles:

$$(2\pi i)^{-1}\int_l=\sum_k \mathrm{res}(w_k)\ [+\mathrm{res}(m_{k+1})]\ .$$

The direct calculation shows the important relation $\mathrm{res}(c_k)=-\mathrm{res}(\tilde{c}_k)$, where $c=v$, w, or m. Thus the integral over the circle contains implicitly the contribution $-\frac{1}{2}\sum_k \mathrm{res}(v_k)+\mathrm{res}(w_k)\ [+\mathrm{res}(m_{k+1})]$, among that from the essential singularity. The conclusion is that $\langle x_t\rangle$ consists of the continuous part given by the second integral in (B1); the discrete part such that the total contribution from each v_k and its counterpart $\tilde{v}_k$ is $-\frac{1}{2}\mathrm{res}(v_k)$; and from each pair $w_k,\tilde{w}_k$ ($m_{k+1},\tilde{m}_{k+1}$) is $+\frac{1}{2}\mathrm{res}(w_k)$ [and $+\frac{1}{2}\mathrm{res}(m_{k+1})$, in case IV]. Equation (4.9) expresses this formulation analytically, and therefore it is valid generally, irrespective of the particular relation between parameters.

Note that the crucial points of the analysis in this and in the preceding section are the same. Namely, we have shown the proper way of replacing some functions—the modified Bessel function I in (A2) and the confluent hypergeometric function ${}_1F_1$ in $\mathcal{I}$—by the associated one, K or ψ (the second solution of Bessel or Kummer differential equation, respectively), without affecting the values of the integrals of interest. Such a procedure has enabled us to interchange the order of integration and carry out the integration over s. The ways of extending the results on the whole parameter space have been analogous.

[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984); P. Hänggi, Z. Phys. B **31**, 407 (1978); J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982); R. Graham and A. Schenzle, *ibid.* **26**, 2676 (1982); F. J. de la Rubia, Phys. Lett. **110A**, 17 (1985); C. R. Doering and W. Horsthemke, J. Stat. Phys. **38**, 763 (1985); C. R. Doering, Phys. Rev. A **34**, 2564 (1986).

[2] C. Van Den Broeck, J. Stat. Phys. **31**, 467 (1983).

[3] A. Fuliński, Phys. Lett. A **126**, 84 (1988); Acta Phys. Pol. A **74**, 193 (1988).

[4] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981); C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[5] J. B. Morton and S. Corssin, J. Stat. Phys. **2**, 153 (1970); cf. also Ref. [6].

[6] R. Phythian and W. D. Curtis, J. Stat. Phys. **42**, 1019 (1986).

[7] A. Morita, J. Chem. Phys. **76**, 4191 (1982); A. Morita and J. Makino, Phys. Rev. A **34**, 1595 (1986); A. Morita, *ibid.* **41**, 754 (1990).

[8] J. J. Brey, C. Aizpuru, and M. Morillo, Physica A **142**, 637 (1987).

[9] M. Suzuki, K. Kaneko, and S. Takesue, Prog. Theor. Phys. **67**, 1676 (1982); M. Suzuki, S. Takesue, and F. Sasagawa, *ibid.* **68**, 98 (1982).

[10] For the problem of reconstruction of $P(x,t)$ from its moments, cf. F. Marchesoni, Z. Phys. B **62**, 505 (1986); R. F. Pawula, Phys. Rev. A **36**, 4996 (1987).

[11] S. Succi and R. Iacono, Phys. Rev. A **36**, 5020 (1987); M. J. Englefield, J. Stat. Phys. **52**, 369 (1988); A. Debosscher, Phys. Rev. A **40**, 3354 (1989); **42**, 4485 (1990); **44**, 908 (1991); **44**, 7929 (1991); M. Kuś, E. Wajnryb, and K. Wódkiewicz, *ibid.* **43**, 4167 (1991); K. J. Phillips, M. R. Young, and S. Singh, *ibid.* **44**, 3239 (1991).

[12] J. Masoliver and G. H. Weiss, Physica A **149**, 395 (1988).

[13] J. M. Sancho and M. San Miguel, Z. Phys. B **36**, 357 (1980); P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **56**, 333 (1984); P. Hänggi, T. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985); P. Jung and H. Risken. Z. Phys. B **61**, 367 (1985); P. Jung and P. Hänggi, Phys. Rev. A **35**, 4464 (1987); S. Faetti and P. Grigolini, *ibid.* **36**, 441 (1987); N. G. van Kampen, J. Stat. Phys. **54**, 1289 (1989); P. Hänggi, P. Jung, and F. Marchesoni, *ibid.* **54**, 1367 (1989); H. Gang and H. Haken, Phys. Rev. A **41**, 7078 (1990); H. Gang, *ibid.* **43**, 700 (1991); A. E. Sitnitsky, Phys. Lett. A **162**, 155 (1992).

[14] See Refs. [1,3,7,8,9,13] and A. Schenzle and H. Brand,

Phys. Rev. A **20**, 1628 (1979); S. Faetti, C. Festa, L. Fronzoni, P. Grigolini, F. Marchesoni, and V. Paleschi, Phys. Lett. **99A**, 25 (1983); C. Festa, L. Fronzoni, P. Grigolini, and F. Marchesoni, *ibid.* **102A**, 95 (1984); F. de Pasquale, J. M. Sancho, M. San Miguel, and P. Tartaglia, Phys. Rev. A **33**, 4360 (1986); P. Jung and P. Hänggi, Phys. Rev. Lett. **61**, 11 (1988); M. R. Young and S. Singh, Phys. Rev. A **38**, 238 (1988); M. C. Torrent and M. San Miguel, *ibid.* **38**, 245 (1988); R. Mannella, C. J. Lambert, N. G. Stocks, and P. V. E. McClintock, *ibid.* **41**, 3016 (1990); G. Debnath, F. Moss, T. Leiber, H. Risken, and F. Marchesoni, *ibid.* **42**, 703 (1990); J. B. Swift, P. C. Hohenberg, and G. Ahlers, *ibid.* **43**, 6572 (1991); E. Fick, M. Fick, and G. Hausmann, *ibid.* **44**, 2469 (1991); T. C. Elston and R. E. Fox, *ibid.* **44**, 8403 (1991); S. Zhu, *ibid.* **45**, 3210 (1992).

[15] The last restriction is not in principle required because appropriate solution of the type (2.2) is available for any $a(t)$ and $b(t)$. It seems, however, that only a strongly limited number of time-dependent (deterministic) coefficients is allowed in order to do the averaging in a closed analytical form (see also Sec. VI).

[16] The so-called transient Mellin function, related to the probability of transition by the inverse Mellin transformation with respect to ω.

[17] P. Hänggi, Z. Phys. B **36**, 271 (1980); **43**, 269 (1981); G. Faraci and A. R. Pennisi, Phys. Rev. A **33**, 583 (1986); J. Masoliver, *ibid.* **35**, 3918 (1987); S. C. Lowen and M. C. Teich, *ibid.* **43**, 4192 (1991).

[18] R. Zygadło, Acta Phys. Pol. A **78**, 277 (1990).

[19] This statement applies to rational functions for which the binomial decompositions of numerator and denominator can be found.

[20] I. L. Luke, *The Special Functions and Their Approximations* (Academic, New York, 1969); I. M. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965); H. Exton, *Handbook of Hypergeometric Integrals* (Horwood, Chichester, 1978).

[21] Frequently it is the indented straight line lying parallel to imaginary axis because of exponential decay of Γ in this direction.

[22] A. Schenzle and H. Brand (Ref. [14]) have solved appropriate time-dependent FPE. The formal expansion for transient moments has been found by *L.* Brenig and N. Banai, Physica D **5**, 208 (1982), using the Carleman imbedding method (see also Ref. [23]), and in Ref. [9] by direct averaging. The comparison with FPE results has been done by C. W. Gardiner and R. Graham, Phys. Rev. A **25**, 1854 (1982).

[23] R. Graham and A. Schenzle, Phys. Rev. A **25**, 1731 (1982).

[24] For $A \to 1^-$ $\varphi(A)$ approaches infinity. It means that the tangent line to $\langle x_t \rangle$ at $t=0^+$ approaches the ordinate axis. The last remains (qualitatively) unchanged when A becomes greater than unity, provided $\mu > 0$. For $\mu < 0$ and $A \geq 1$ the mean value does not exist.

[25] It must appear also for some $x_0 < x_{st}$. To get the precise description, the condition $\langle \dot{x}_t \rangle = 0$ should be solved. Unfortunately, it seems that this cannot be simply done.

[26] Compare, e.g., J. J. Brey, J. M. Casado, and M. Morillo, Z. Phys. B **66**, 263 (1986); A. Fuliński and T. Telejko, Phys. Lett. A **125**, 11 (1991); R. Benzi, S. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); **18**, 2239 (1985); B. McNamara, K. Wiesenfeld, and P. Roy, Phys. Rev. Lett. **60**, 2626 (1988); L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, *ibid.* **62**, 349 (1989); P. Jung and P. Hänggi, Europhys. Lett. **8**, 505 (1989); R. F. Fox, Phys. Rev. A **39**, 4148 (1989); C. Presilla, F. Marchesoni, and L. Gammaitoni, *ibid.* **40**, 2105 (1989); G. Debnath, T. Zhou, and F. Moss, *ibid.* **39**, 4323 (1989); T. Zhou, F. Moss, and P. Jung, *ibid.* **42**, 3131 (1990); P. Jung and P. Hänggi, *ibid.* **44**, 8032 (1991); L. Gammaitoni, F. Marchesoni, M. Martinelli, L. Pardi, and S. Santucci, Phys. Lett. A **158**, 489 (1991); F. Moss, Ber. Bunsenges. Phys. Chem. **95**, 371 (1991).

[27] Note that these noises appear as a result of the appropriate symmetric or asymmetric limit of (colored) dichotomous Markov process, respectively; see Ref. [2].

[28] $P_{st}(x)$ (6.2) exhibits the so-called noise-induced transitions. The transitions at points $A = a$ and 0 are related to the change of the analytical formulas describing the moments: (4.10) $\leftrightarrow$ (4.6) and (4.6) $\leftrightarrow$ (4.11), respectively. The second one ("near" the deterministic case) corresponds also to the discontinuous change of the support of probability density distribution: $(0,x_{st}) \to \{x_{st}\} \to (x_{st},\infty)$, where $x_{st} = (b/a)^{-1/\mu}$ is the deterministic stationary state [see also J. M. Sancho, M. San Miguel, L. Pesquera, and M. A. Rodriguez, Physica A **142**, 522 (1987)]. However, the moments change continuously (even C^∞) during these transitions. The last follows from the asymptotic properties of Γ function.

[29] The invariance of the Whittaker function with respect to the sign of its second parameter is to be noted.